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Fractal dimension and Quark–Gluon plasma

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The properties of quark–gluon plasma (QGP) have been studied considering it as fractal fluid where configuration space shows a noninteger dimension. The fractal dimension for the QGP droplet is suggested. The thermodynamic properties like radial distribution function, packing fraction, excess entropy and susceptibility are studied. The viscosity of the QGP phase has also been investigated and fluid is found to behave like a superfluid. Some interesting observations are made.

Keywords: QGP; fractal fluid; fractal dimension; viscosity.

1. Introduction

A new phase of matter has been suggested to exist where the nuclear matter of neutron and proton transforms into deconfined phase of quark–gluon plasma (QGP) with the discovery of asymptotic freedom in quantum chromodynamics (QCD). This new phase of matter existed in the early stage of the Universe and subsequently expanded hydrodynamically to form colorless matter and hadronization started. In the experimental sectors many relativistic high energy collision experiments hint at the formation of QGP.[1–4] The state is realized as the state of distribution of quarks and gluons with no color confinement in the weakly coupled state. The QGP signatures include primarily large multiplicity fluctuation in small rapidity interval,[5] suppression of heavy mesons[6,7] and strangeness enhancement.[8] Major aim of relativistic heavy ion collision is to study

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QGP phase which is basically understanding of QCD matter at the most basic level. QGP phenomenon was observed at the relativistic heavy ion collider (RHIC) that was carried over to the large hadron collider (LHC) which investigates QCD matter at high energy and high density. The experiments provide unique opportunity to study the nonequilibrium dynamics of deconfined QCD matter (QGP), microscopic properties of QGP and also exploring the structure of nucleons. Dynamical fluctuations in relativistic nuclear collision may be considered as an important signature of QGP formation. It is observed that the fluctuation shows the scaling behavior.[9] The scaling behavior of factorial moment of multiparticle production hints at the presence of intermittency which may be attributed to the fractal nature of particle production.[10,11] High energy collision indicates the formation of a system with very short-lived thermodynamic equilibrium. The formation of such short-lived fire ball which reaches equilibrium so fast needs explanation. Hagedron[12] proposed a self-consistent thermodynamics (SCT) which is able to explain several features of hot hadronic system. The asymptotic statistical bootstrap leads to the self-consistent asymptotic mass spectrum which grows as $\rho(m) \sim \text{const.}\, m^{\frac{-5}{2}} e^{\frac{m}{T_0}}$ as $m \to$ infinity where T_0 is the highest possible temperature for strong interaction. The mass spectrum shows a power law behavior with exponent $(-5/2)$. QGP as a fractal fluid is one of the interesting areas of research and studied by a number of authors. Many growth processes and aggregation of liquid drop can be described by the concept of fractal geometry. Heinen *et al.*[13] developed a theory for the fractal analogue of the simplest classical generic hard core fluid model in integer dimension between $d = 1$ and 3 to fractal dimension in the range $1 \leq d \leq 3$. They considered fractal particles in a fractal configuration space for noninteger dimensions and observed that fractal liquids can described as microphase separated by binary liquids in porous media and highly branched liquid droplets confined to a fractal polymer backbone in a gel. Santos *et al.*[14] discussed the equation of states and radial distribution function of hard-core and hard sphere fluid in fractal dimension between $1 \leq d \leq 3$. It is suggested that the QGP behaves like perfect fluid and can be studied by ideal hydrodynamics which can be well described by strongly coupled liquid than the weakly coupled gaseous state. There are extensive studies on the pattern of multiplicity distribution in hadron colliders which show power law distributions and demand the hydrodynamic description of the evolution of the system.[10] Recently Li *et al.*[15] made an excellent review on intermittency in heavy ion collisions and power law fluctuation in QCD phase diagram in connection with fractality, self-similarity and intermittency. Experimental exploration of intermittency and fractality in high energy collisions by NA49, NA61, STAR is discussed in detail with phenomenological models. The study of scaled factorial moment in transverse momentum space is crucial and it is reported that NA49 and STAR experiments show a good power law behavior of correlator indicating intermittency and fractality in collisions.

Describing QGP as fractal fluid is one of the interesting areas of understanding the QGP dynamics and has been studied by a number of authors.[11,12] Antoniou *et al.*[16] studied the QCD phase transition and QGP in 3D Ising model. They also studied the fractal properties of QGP associated with the fractal dimension of clusters. It has been observed that for a minimal volume, the self-similarity of fireball evolution breaks down and density–density correlation obeys a power law. Endrodi *et al.*[17] studied the fractal structure of the center of the strongly interacting matter near deconfining Polyakov loops and have observed the fractal dimension to vary from 2.5 to 2.1 in the range $T = 0.75$–$2\ T_c$. They pointed out that the fractal structure is the indication of small shear viscosity and large opacity of QGP phase whereas Suzuki *et al.*[18] studied the fractal dimension of high energy nucleus–nucleus collision as QGP signature and suggested the fractal dimension as 1.5 $(1 < d < 2)$ for different set of parameters. Dimensionality of a system is a very important property which governs the dynamics of the system itself. The basic characteristics of a fractal fluid is noninteger mass dimensionality of the distribution. Tarasov[19] discussed the noninteger dimensional approach for fractal fluid and developed a formalism to study the flow with Navier–Stokes equation for noninteger dimensional space for fractal fluid. Kalaydzhyan[20] pointed out that QGP can be considered as chiral superfluid with normal component as the thermalized matter whereas the superfluid part consists of long wavelength fermionic states moving independently. Deppman *et al.*[21] made a wonderful review on the recent observations of power law distribution of the particles in high energy collisions (LHC) with thermodynamic description of the system and reviewed the understanding of the fractal structure present in Yang–Mills fields. They pointed out that the interpretation of power law distribution by thermofractal structure may have possibility to explain structure and properties of QCD phase diagram with the connection of such distribution with fractal dimension of hadrons.

In this work, we have described QGP as a fluid and studied the thermodynamic properties of QGP droplet. The formation and dynamics of QGP droplet are complicated phenomena and many models have been suggested to get insights into this physical process of droplet formation.[4–8] It is well known that the dimensionality plays an important role in describing the properties of a thermodynamic system. In our previous works we have suggested that hadrons are fractal objects and obtained a fractal dimension of a hadron as $9/2$.[22,23] We have suggested that the QGP fire ball possesses self-similar structure as suggested by Hagedron[12] and it possesses fractal dimension as that of a hadron. We have studied the thermodynamic properties like excess entropy, packing fractions, radial distribution functions and susceptibility. These properties are greatly dependent on the dimension of the system.[13,14] We have also studied the viscosity of QGP fluid and it is observed that the effective viscosity is less than the original viscosity indicating the superfluid nature of the QGP phase.

2. Formulation: Fractal Dimension

The dimensionality of a system is a very important property and nondifferentiability of space-time governs the dimension of the system and may be fractal. It has been suggested that the scale relativity is much more fundamental and is considered to manifest the fact that the set has structure at every minute level. The scale relativity deals with the passage from differential to nondifferential geometry.[24] Nottale[25] pointed out that the resolution at which a system is observed should be considered with its motion at the same footing. The path of the quantum particle is fractal and uncertainty is basically associated with the space-time structure of the quantum particles. While probing any measure N through any infinitesimal resolution scale ξ the fractal dimension (d) of it can be written as

$$N(\xi) \sim \xi^{-d}. \tag{1}$$

$N(\xi)$ is the number of spheres of radius ξ needed to cover the system. The N-hypervolume can be expressed as[26]

$$V(\xi) \sim \xi^{D_T - d}, \tag{2}$$

where D_T is topological dimension and d is fractal dimension. In the context of the statistical model description, a hadron[22,23] is described as an extended system which contains not only a real valence quark–antiquark pair but also an infinite number of virtual $q\bar{q}$ pairs commonly called virtual sea. Pertaining local color symmetry at each point of the color neutral hadron, the number density of quarks and antiquarks is the same for a particular point and then equated to the probability density $|\Psi(r)|^2$ of a hadron. It can be expressed as

$$n(r) = |\Psi(r)|^2 = Ar_0^{-9/2}(r_0 - r)^{3/2}\theta(r_0 - r), \tag{3}$$

where A is a constant and r_0 is the characteristics radius of a hadron. The wave function has a remarkable feature that it is nonanalytic in nature and shows a scaling property.[23] With $\xi = (r_0 - r)$, the above equation can be recast as

$$n(\xi) \sim \xi^{-3}\left(\frac{\xi}{r_0}^{9/2}\right)\theta(\xi). \tag{4}$$

So, the hypervolume takes the form

$$V(\xi) \sim \xi^{3-\frac{9}{2}}. \tag{5}$$

Comparing Eqs. (2) and (5) it is suggested that the fractal dimension d of the hadron comes out to be 9/2. Hagedron[12] pointed out that the mass evolution of QGP fireball varies as $m^{\frac{-5}{2}}$ which is fraction and possesses self-similar structure with a highest possible temperature. QGP is formed in the high energy nucleus–nucleus collisions and statistical fluctuations are observed in particle distribution. We have suggested that the fractal dimension of the QGP fireball to be as that of hadron considering the self-similar structure of QGP produced in high energy collisions. In our subsequent analysis we will study the effect of fractal dimension on the evolution of QGP considering it as a fractal fluid with dimension 9/2.

3. Thermodynamic Properties of QGP

Heinen *et al.*[13] and Santos *et al.*[14] studied the thermostatistical properties of fractal particles in fractal configuration space where particles are fractals and move in a fractal space. They developed equation of states, radial distribution of hard core fractal fluid between one and three dimensions. All the thermodynamic parameters depend on the dimension. The packing fraction (ϕ) of hard-sphere fluids in d dimensions with density ρ can be expressed as

$$\phi = \left(\frac{\pi}{4}\right)^{\frac{d}{2}} \cdot \rho \cdot \sigma^d / \Gamma\left(1 + \frac{d}{2}\right), \tag{6}$$

where σ is the diameter of the hard sphere. With d equal to 9/2 for QGP system as suggested by us, we have obtained

$$\phi = 0.316\rho \tag{7}$$

with $\sigma = 6\,\text{mb} = 0.6\,\text{fm}^2$. Packing fraction is linear function of the density (in unit of nuclear matter density ($\rho = \rho_0$)). It is interesting to note that packing fraction becomes more than 1 at a density ~ 3.4 of density indicating the fact that the system becomes unstable at $\rho \sim 3.4\rho_0$ and maximum density of QGP is 3.4 times of normal nuclear matter density ρ_0.

The radial distribution function runs as

$$g_c\phi = 1 - k_d\phi/(1-\phi)^2, \tag{8}$$

$$g_c\phi = (1 - 2.601\phi)/(1-\phi)^2, \tag{9}$$

where $k_d = [(5-d)(2-d)/4 + (3-d)(d-1)k_2]$ and $k_2 \simeq \frac{2\sqrt{3}}{\pi}$.[14] d is the fractal dimension. We have plotted the variation of radial distribution function with packing fraction ϕ and displayed it in Fig. 1. The distribution function shows a discontinuity at $\phi > 1$.

Excess entropy is the difference in entropies between liquid and ideal gas. It is negative. Density and entropy are closely connected to the structure and dynamics of

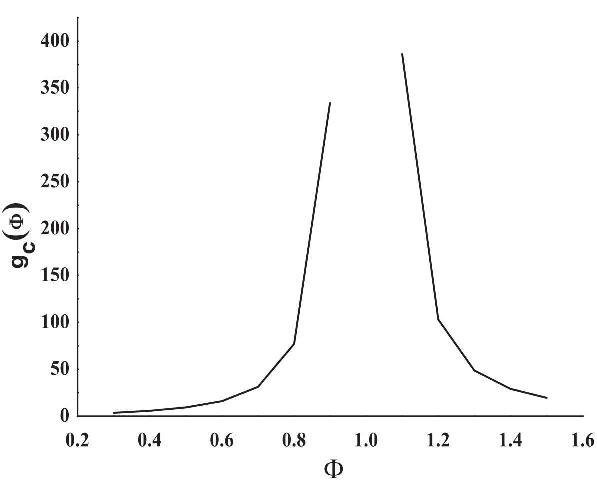

Fig. 1. Packing fraction versus radial distribution function.

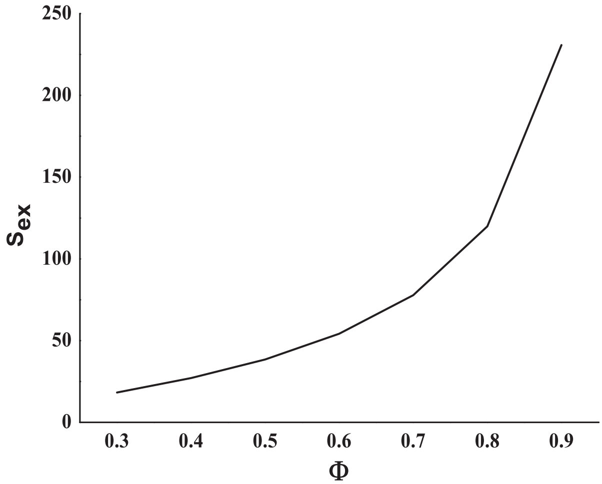

Fig. 2. Packing fraction versus excess entropy.

the fluid state and depends on the dimension of the system. The expression of excess entropy runs as[14]

$$S_{\text{ex}}\phi = -2^{d-1}[1 - k_d\phi/(1-\phi) - k_d \ln(1-\phi)]. \tag{10}$$

We have estimated excess entropy with packing fraction with fractal dimension $d = 9/2$ and displayed in Fig. 2. S_{ex} shows a sharp increase with ϕ.

Isothermal susceptibility is another important parameter for a thermodynamic system and the expression runs as[14]

$$\lambda_T\phi = \left[1 + 2^{d-1}\cdot\phi\cdot\frac{2 - k_d\cdot\phi(3-\phi)}{(1-\phi)^3}\right]^{-1}. \tag{11}$$

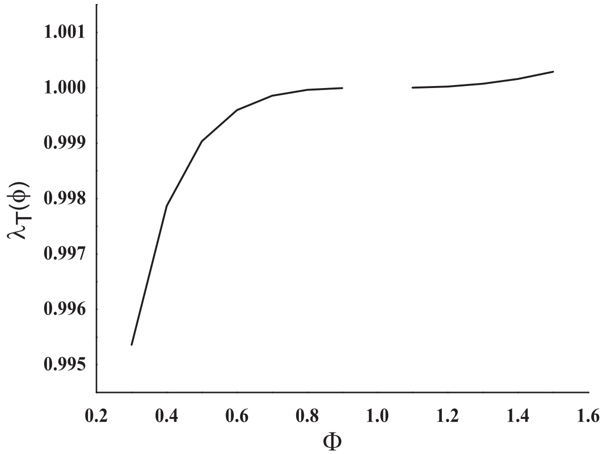

Fig. 3. Packing fraction versus isothermal susceptibility.

We have estimated the isothermal susceptibility $\lambda_T \phi$ and displayed it in Fig. 3 with $d = 9/2$ as suggested. It is interesting to observe that the susceptibility attends a saturation at packing fraction $\phi > 0.8$.

4. Viscosity

The fractal fluid is characterized by the noninteger dimension of the constituents and their power law behavior. The transition from continuum space-time to noninteger dimensional space is studied by number of authors.[19,23] Viscosity of a fractal fluid depends on the dimension of the space. For evaluation of the viscosity of fractal fluid, integration in d-dimensional space has to be considered. Tarasov[19] developed Navier–Stokes equation in noninteger dimensional space and found an expression for the effective viscosity (μ_{eff}) of a fractal fluid flowing through a pipe, which runs as

$$\mu_{\mathrm{eff}} = \left(\frac{D_F + 1}{4}\right) \pi^{(3-D_F)/2} \Gamma\left[\frac{(D_F - 1)}{2}\right] \mu, \tag{12}$$

where (μ) is the normal fluid viscosity and (μ_{eff}) is the effective fluid viscosity in noninteger dimensional space. The mass of the fractal liquid flowing per unit time is obtained as

$$Q = -\frac{\rho\pi}{8\mu_{\mathrm{eff}}} \left(\frac{dp}{dx}\right) R^{D_F+1}, \tag{13}$$

where ρ is the density of the fluid and $\frac{dp}{dx}$ is the pressure gradient of the fluid, R is the radius of the pipe through which the fluid is flowing.

Assuming the fractal characteristics of QGP and with the input of the fractal dimension of QGP, i.e., $(D_F) = 9/2$, we arrive from Eq. (11)

$$\mu_{\mathrm{eff}} = 0.5354\mu. \tag{14}$$

Subsequently Eq. (12) also appears to be

$$Q = -0.73\left(\frac{\rho}{\mu}\right)\left(\frac{dp}{dx}\right) R^{\frac{11}{2}}. \tag{15}$$

The effective fluid viscosity μ_{eff} is found to be less than the normal fluid viscosity μ indicating the superfluid nature of the QGP. The superfluidity nature of the QGP strongly depends on the fractal nature of QGP.

5. Conclusion

In this work we have studied the properties of fractal fluid in the framework of thermostatistical system. We have investigated radial distribution function, excess entropy and susceptibility as a function of packing fraction. All these thermodynamic quantities depend on the dimension of the system. Packing fraction is very important parameter which is related to the stability of system. It depends on the density i.e., number of particles per volume. It may be mentioned that the packing fraction may be positive, negative and the system is stable when it is negative and unstable when positive. With dimension of the system as 9/2 as suggested by us and with a typical scattering cross-section of QGP formation as 0.6 mb, the packing fraction is found to be $0.316\rho_0$ where ρ_0 is normal nuclear matter density. From expression (7), we have observed that up to a density of $\rho = 2.4\rho_0$, the QGP behaves like closely packed system and above it may turn to a multiple component system with mixture of liquid and gas. Radial distribution function has been estimated and displayed in Fig. 1. Radial distribution function strongly depends on the properties of the system and related to bulk density to local density. From Fig. 1, it is found that the peak of the $g_c(r)$ occurs at $\phi > 0.8$ which hints at the fact that the maximum packing and co-ordination may be at $\phi \simeq 0.8$. It is interesting to note that there is a Λ type phase transition from QGP phase at $\phi \simeq 0.8$. For $\phi < 1$, a sharp increase in radial distribution is observed and it is found to decrease sharply for packing fraction more than one. QGP phase becomes unstable and will tend to disintegrate via other processes.

Figure 2 displays the excess entropy S_{ex} as a function of packing fraction ϕ. Excess entropy is a critical quantity which represents the density anomaly in the fluid and turns out to be a sensitive indicator of structural and dynamical changes that take place in the system. It is interesting to observe that with fractal dimension 9/2, the S_{ex}–ϕ graph shows almost a linear behavior till $\phi < 0.70$ and a sharp increase above $\phi > 0.70$ indicating an approach to more disordered phase and a closed packed multiple component structure. S_{ex} also represents reduced viscosity and diffusivity. It is pertinent to point out in this context that by increasing the packing fraction beyond 0.74, the systems behave like a jammed packing system with structural complexity of stochastic process.

Figure 3 displays our study of the isothermal susceptibility (compressibility) $\lambda_T\phi$ with packing fraction ϕ. It is observed that the λ_T increases sharply with ϕ and reaches saturation at $\phi \simeq 0.80$ indicating the fact that QGP becomes highly incompressible beyond $\phi \simeq 0.80$ with a singularity at $\phi = 1$. A slow increase $\lambda_T\phi$ is observed beyond $\phi \simeq 1.2$.

We have also studied the viscosity of QGP fluid and effective dynamic viscosity μ_{eff} which is found to be less than normal viscosity μ. Velkovaska[27] made an excellent review on properties of QGP matter and pointed out that it behaves like a perfect fluid with negligible viscosity. The bulk medium production in high energy collisions displays hadron momentum spectra, angular distribution like flow patterns and particle yield ratio. The most exciting discovery of RHIC is that the new medium shows strong collective dynamics and the system is well described by ideal hydrodynamics quantitatively. Self-similarity is a salient feature if fireball formation of QGP is considered. Hadron mass distribution has been described by power law evolution as $\rho(m) = Km^{-5/2}$ by Hagedron.[12] We have suggested that the hadron possesses a fractal dimension 9/2. Deppman *et al.*[21] pointed out that the power law distribution of particles in high energy experiments may be connected to the fractal dimension for the hadrons. They studied the fractal structure of Yang–Mills field and studied the evolution as thermofractal with Tsallis statistics entropy index. Megias *et al.*[28] have also investigated application of Tsallis statistics to high energy hadron production including hadron model, hadron mass spectrum, QCD and neutron star. They have suggested the structure of self-similar thermodynamical system with fractal behavior leads to Tsallis nonextensive statistics. Recently Zborovsky *et al.*[29] studied self-similarity and fractality of hadron production. The z-scaling in high p_T particle production is found to possess power law behavior, fractality and self-similarity. They have suggested that the fractal dimension is a quantity which has quantum nature and z-scaling shows fractality of hadron production at constituent level.

This work focusses on the study of QGP fire ball as a fractal fluid which possesses self-similar structure. The fractal dimension is higher as predicted in statistical model which is in the range ($3 < D_F < 9/2$). The nature of fluid may be suggested to be super fractal.[30] Thermodynamic properties of the system are studied. It is pertinent to point out here that the large fluctuation in rapidity distribution is signature of the QGP and larger the value of D_F, higher is the fluctuation. QGP is suggested to behave as hard sphere liquid with fractal dimension $9/2$. The packing fraction ϕ gives the interesting information for the QGP thermodynamics in the range $\phi \simeq 0.70$–0.85. The maximum QGP packing fraction is suggested to be 3.4 times of normal nuclear density ρ_0. The perfect fluid behavior indicates that the QGP is strongly coupled system in spite of the fact that it is short-lived. Study of QGP dynamics as hard sphere fluid is an emerging field of research. It may be mentioned that in the current investigation, the analysis is done in configuration space. Antoniou *et al.*[31] investigated baryon number density fluctuation near QCD critical intermittency which shows power law behavior in transverse momentum spectra. They observed that factorial moment shows a power law behavior implying a large density fluctuation at critical point due to fractal geometry. In our future work we will extend the methodology to investigate the thermodynamic properties in momentum space through Fourier transformation. With the avant-garde of new improved experiments, the detailed study of QGP dynamics is underway which aims to reveal exact properties of QGP and its evolution. It is now the key question of how to probe the nature and properties of this short-lived QGP produced at the early time of heavy ion collisions.

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